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The steady Navier–Stokes/energy system with temperature-dependent viscosity—Part 1: Analysis of the continuous problem

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SUMMARY

In this first part we propose and analyse a model for the study of two-dimensional incompressible Navier–Stokes equations with a temperature-dependent viscosity. The flow is supposed in a mixed convection regime and considers an outflow region, leading to a strongly coupled problem between the Navier–Stokes and energy equations, which will be justified theoretically. The coupling in the continuous problem is treated by an outer temperature fixed point strategy. Existence results for a particular variational formulation follows from this study. Further, a particular uniqueness result for small data is also obtained. Copyright © 2007 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Often in Newtonian fluid flow analysis, the viscosity property is considered just as a constant parameter. This strong simplification for the model of flow motion may be justified for isothermal flows or very small temperature differences. Other reason of this simplification is associated sometimes with the computational work in the numerical simulations (further additional loops in the resolution strategy). Nevertheless, a closer view for the thermophysical properties of a simple

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fluid like water, for instance, shows that the variations for the dynamic viscosity can reach 250% for normal temperature range (see [1]).

Besides, there exists a robust mathematical theory for the incompressible 'constant viscosity' Navier–Stokes equations (see [2–5]), which can be generalized in some particular situations to the variable viscosity case (see, for instance, [6, 7]). For instance, if the problem has only Dirichlet data, the extension to the non-constant physical coefficients case is straightforward provided that these physical coefficients are uniformly bounded. Even if there are different fragments with 'natural' (Neumann) and 'essential' (Dirichlet) boundary conditions (BCs), the analysis is also extensible from the classical Dirichlet case provided that the region for the natural BCs are complementary in the two variables (velocity and temperature, as in [8]). There exist also situations in which the prescription of fluxes or pressure drops at the outflow allows a mathematical analysis (see, for instance, [9, 10]).

However, for open, developing flows, the standard BCs for velocity and temperature consider the *same* outflow region, in which engineers and physicists consider often 'natural' outflow BCs at this portion, also called 'do nothing' BCs. This physical consideration does not match in general with any of the three situations considered above, and the mathematical analysis becomes much more difficult. In addition to the 'non-Laplacian' formulation, consequence of the non-constant viscosity, the theoretical tools such as the lifting, the *a priori* bounds and the regularity results are not trivial.

In this work, our interest is to take into account the coupling due to viscosity variations with temperature, the buoyancy effect and outflow BCs for flow situations such as flow in channels or ducts. For this, we shall deduce a physical model and propose a variational formulation which can be used in this framework, being closer to the natural variational formulation of the set of equations related to the physical model. In this first part, we will deduce and justify the existence and uniqueness for this steady variational problem which considers some generalized outflow BCs, technique that was first proposed in [11] for the isothermal case, but that we implement and analyse in the non-isothermal coupled equations. We show also that uniqueness of the coupled problem is allowed for moderate Reynolds and Péclet numbers and small data.

2. A BRIEF BIBLIOGRAPHICAL REVIEW

We shall discuss some relevant bibliographic references in order to raise the differences with our case study.

Among the first references concerning the analysis of the Navier–Stokes equations coupled with the energy equation through a viscosity coupling, which is also one of the main references in our work, we refer to [12], who analyse existence of solutions for Bingham fluids with temperature-dependent viscosity. The main references concerning the use of generalized outflow BCs implemented in this work are [11, 13], where the incompressible and compressible Navier–Stokes equations without the coupling with the energy equation were analysed.

In [14] the unsteady problem in compressible fluids is analysed but, as in [12], only Dirichlet BCs are considered. More recently, in [15] one can find general strategies for different kind of pressure–velocity–temperature couplings. A general steady non-linear temperature-dependent diffusion coupling is made in [6], from which we follow the decoupling strategy. They consider homogeneous Dirichlet BCs only. In [16] an existence result for a steady problem with temperature-dependent viscosity is proved based on monotone operators. Boussinesq models with constant thermophysical

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properties are considered in [8, 17–19], in which existence and some regularity results are shown. The outflow hypothesis $\mathbf{u}\cdot\mathbf{n}\geqslant 0$ at the exit part of the domain is taken by these authors for open-channel flows.

Finally, concerning outflow BCs, a general introduction and rigorous mathematical analysis is made in [10]. The consideration of the negative part of the velocity flux in the variational formulation was introduced in [11, 13]. Other admissible outflow BCs are mentioned in [9, 20, 21].

3. PHYSICAL MODEL, GOVERNING EQUATIONS

Let Ω be a two-dimensional domain occupied by an incompressible Newtonian fluid. Even if the mathematical analysis presented in the next section is only valid for the two-dimensional case, the model remains valid in the more realistic three-dimensional situation.

3.1. Physical model

The general governing equations for a Newtonian incompressible fluid are the Navier–Stokes, continuity and energy equations (see, for instance, [22])

$$\rho\left(\frac{\partial}{\partial t}\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right)=\widetilde{\mathbf{F}}+\rho\mathbf{g}-\nabla p+\nabla\cdot(\mu D(\mathbf{u})) \tag{1}$$

$$\nabla\cdot(\rho\mathbf{u})=0 \tag{2}$$

$$\rho c_p\left(\frac{\partial}{\partial t}T+\mathbf{u}\cdot\nabla T\right)=\nabla\cdot(\kappa\nabla T) \tag{3}$$

Here, we write $\mathbf{u}$ as the velocity field, p the pressure, ρ the density, T the temperature, μ the dynamic viscosity, κ the thermal conductivity, c_p the specific heat, $\mathbf{g}$ the gravity vector force, t the time and $D(\mathbf{u})$ the deformation tensor, symmetric part of $\nabla\mathbf{u}$. We write formally as $\widetilde{\mathbf{F}}$ the action of external forces.

In our work, we will permit the coefficients μ, κ and ρ to be dependent on the temperature T (see Equations (4)–(6)).

Most of the fluids, particularly liquids, are in fact very weakly compressible. Following Schlichting (cf. [23, p. 9]), compressibility can be neglected if $\frac{1}{2}M^2\ll 1$, with M being the Mach number.

Thus, for weak variations of temperature around some reference value T_m, and neglecting the pressure variations under the low-Mach assumptions, we can state as approximation of ρ

$$\rho\approx\rho_m[1-\beta(T-T_m)]$$

where ρ_m is the density value associated with this reference temperature T_m. We also define the values $\mu_m=\mu(T_m)$ and $\kappa_m=\kappa(T_m)$ as the dynamic viscosity and thermal conductivity values for this reference temperature. We also define a kinematic viscosity value as $\nu_m=\mu_m/\rho_m$. The constant β is the thermodynamic dilatation coefficient, defined by $\beta=(1/\rho)\partial\rho/\partial T$ at constant pressure.

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Under the Boussinesq approximation (see [24, p. 119] for a discussion on this matter), Equations (1)–(3) become

$$\rho_m\left(\frac{\partial}{\partial t}\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right)-\nabla\cdot(\mu(T)D(\mathbf{u}))+\nabla p=\mathbf{F}-\mathbf{g}\rho_m\beta(T-T_m) \tag{4}$$

$$\nabla\cdot\mathbf{u}=0 \tag{5}$$

$$\rho_m c_p\left(\frac{\partial}{\partial t}T+\mathbf{u}\cdot\nabla T\right)-\nabla\cdot(\kappa(T)\nabla T)=0 \tag{6}$$

Next, we choose in addition to the reference temperature, a reference velocity $\mathbf{u}_m$ and a reference length H_m. We keep the time scale assuming implicitly that $t_m=H_m/\mathbf{u}_m$.

We divide Equation (4) by ρ_m and (6) by $\rho_m c_p$, and introduce the functions

$$\mu^*(T)=\frac{\mu(T)}{\mu_m},\quad \kappa^*(T)=\frac{\kappa(T)}{\kappa_m}$$

With these functions, Equations (4)–(6) can be written as

$$\frac{\partial}{\partial t}\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}-\left[\frac{\mu_m}{\rho_m}\right]\nabla\cdot(\mu^*(T)D(\mathbf{u}))+\left[\frac{1}{\rho_m}\right]\nabla p=\left[\frac{1}{\rho_m}\right]\mathbf{F}-\left[\frac{1}{\rho_m}\right]\mathbf{g}\rho_m\beta(T-T_m) \tag{7}$$

$$\nabla\cdot\mathbf{u}=0 \tag{8}$$

$$\frac{\partial}{\partial t}T+\mathbf{u}\cdot\nabla T-\left[\frac{\kappa_m}{\rho_m c_p}\right]\nabla\cdot(\kappa^*(T)\nabla T)=0 \tag{9}$$

In order to obtain a non-dimensional model, we define $\mathbf{x}^*=\mathbf{x}/H_m$, $\mathbf{u}^*=\mathbf{u}/|\mathbf{u}_m|$, $p^*=p/(\rho_m|\mathbf{u}_m|^2)$ (dynamic pressure) and $T^*=(T-T_m)/\Delta T$, with ΔT being a characteristic temperature difference. With the above reference parameters, we introduce a *reference* Reynolds, Grashof and Péclet number associated with T_m as

$$Re_m=\frac{|\mathbf{u}_m|H_m}{\nu_m},\quad Gr_m=\frac{|\mathbf{g}|\beta\Delta T H_m^3}{\nu_m^2},\quad Pe_m=\frac{\rho_m c_p|\mathbf{u}_m|H_m}{\kappa_m}$$

The consideration of these non-dimensional variables and parameters in (7)–(9) gives the following non-dimensional model, in which the asterisks have been omitted for simplicity (for a detailed analysis, see [25])

$$\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}-\frac{1}{Re_m}\nabla\cdot(\mu(T)D(\mathbf{u}))+\nabla p=\frac{Gr_m}{Re_m^2}T\mathbf{k}+\mathbf{F} \tag{10}$$

$$\nabla\cdot\mathbf{u}=0 \tag{11}$$

$$\frac{\partial T}{\partial t}+\mathbf{u}\cdot\nabla T-\frac{1}{Pe_m}\nabla\cdot(\kappa(T)\nabla T)=0 \tag{12}$$

In (10), $\mathbf{k}$ is the unitary vector acting in the opposite direction to gravity. For convenience, we keep the expression $\mathbf{F}$ to refer the external forces (now dimensionless). In general, the bidimensional validity of the model is guaranteed when $Gr\ll Re^2$.

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In order to perform the mathematical analysis, it remains only to specify the properties of the viscosity and conductivity functions, and state the BCs.

The values of the dynamic viscosity in liquids are usually approximated by exponential correlations (see [26–28]). Among these approximations, the most common viscosity laws for the approximation of the viscosity are the Arrhenius and Andrade (also called Nahme) laws, expressed here in dimensional form, valid for Kelvin degrees

$$\text{Arrhenius Law:}\ \mu(T) = C_1 \exp^{C_2/(C_3+T)} \tag{13}$$

$$\text{Andrade's Law:}\ \mu(T) = C_1 \exp^{C_2/T} \tag{14}$$

For water in a range of [10–100°], the values for these constants are obtained usually by least-squares fitting. Thus, for the Andrade law, the values are $C_1 = \exp(-12.9896)$ and $C_2 = 1780.622$ (see [28]). It is clear that the validity of these correlation formulas depends on the range of temperature to be considered. The use of an additional constant in the Arrhenius Law allows more precision than Andrade Law, which can be of interest in other liquids. For water, Andrade's Law gives good accurate values in small ranges of temperature, such as [10–100°]. For these values of temperature, the thermal conductivity is, in general, poorly influenced by temperature, and this small influence can be normally approximated with accuracy by linear correlations or simply assumed as constant.

For gases, the most utilized correlation formula for the dynamic viscosity is given by the Sutherland's law:

$$\text{Sutherland's Law:}\ \mu(T) = \mu_m \left(\frac{T}{T_m}\right) \frac{T_m + C_1}{T + C_2} \tag{15}$$

In (15), and for a gas like air, $T_m = 273\,\text{K}$, $\mu_m = \mu(T_m)$, $C_1 = C_2 = 110.5\,\text{K}$. This formula suffers a degradation in the quality of the approximation far away from the reference temperature, forcing to another least-squares fitting around another reference temperature for keeping the validity (see [25]).

Further, in the case of gases, the dynamic viscosity and the thermal conductivity present a similar behaviour (see [29]), as shown by the relation

$$\kappa(T) = \frac{\mu(T) c_p}{Pr} \tag{16}$$

where Pr is the Prandtl number. In order to perform the mathematical analysis valid for these two kinds of fluids (liquids and gases), we will take into account only the common properties for the dynamic viscosity and thermal conductivity functions for liquids and gases, that is: the dynamic viscosity and thermal conductivity viewed as functions of the temperature are strictly positive, Lipschitz continuous, and uniformly bounded functions, i.e. there exist positive constants μ_1, μ_2 and κ_1, κ_2 such that, in the range of validity of the formulas, we have

$$\mu_1 \leqslant \mu(T) \leqslant \mu_2, \quad \kappa_1 \leqslant \kappa(T) \leqslant \kappa_2 \tag{17}$$

3.2. Boundary conditions

This study focuses on open flows, in particular, straight channels. This interest allows to identify the behaviour of the fluid flow due only to the temperature variations, but the following analysis

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applies to other domains too. The effect of the geometry in the constant-property case was analysed numerically in [30] and the references therein.

Concerning BCs, at the entry we must choose a regular velocity profile, such as parabolic (Poiseuille), and a prescribed value T_e for the temperature. At the walls, no-slip conditions for velocity and prescribed temperature distributions $T_u(\mathbf{x})$ and $T_l(\mathbf{x})$ for the upper and lower walls, respectively. The entry and the walls are in consequence associated with Dirichlet BCs. In the outflow portion of the domain, the BCs to prescribe are not so clear. There is not a physical boundary in the case of open flows, but some kind of BCs are needed in order to state the mathematical problem. For this kind of flow, a frequently used assumption as temperature outflow BCs is to state a *zero flux density*, which in this case is expressed by $\kappa(T)\nabla(T)\cdot\mathbf{n}=0$. As far as the thermal conductivity is strictly positive, this is equivalent to the classical outflow condition $\partial T/\partial\mathbf{n}=0$ used in many numerical simulations. For the velocity field, it seems more appropriate to state that the outflow *zero normal traction forces*, that is, $\boldsymbol{\sigma}(\mathbf{u}, p):\mathbf{n}=0$, with $\boldsymbol{\sigma}$ being the stress tensor, which depends here on the temperature through the viscosity: $\boldsymbol{\sigma}(\mathbf{u}, p)=-p\boldsymbol{\delta}+(1/Re_m)\mu(T)D(\mathbf{u})$, where $\boldsymbol{\delta}$ is the Kronecker's tensor. We note that even if the viscosity function is also strictly positive, we have not *a priori* the equivalence with the classical (numerical) outflow condition $\partial\mathbf{u}/\partial\mathbf{n}+p\mathbf{n}=\mathbf{0}$ used in many numerical simulations of constant-property fluid flows (see [21]).

The BCs are summarized as follows:

$$\begin{aligned} \mathbf{u}=\mathbf{u}_e(\mathbf{x}), &\quad \text{at the inflow}, \quad T=T_e(\mathbf{x}), \quad \text{at the inflow}\\ \mathbf{u}=\mathbf{0}, &\quad \text{at walls}, \quad T=T_u(\mathbf{x}), T_l(\mathbf{x}), \quad \text{at walls}\\ \boldsymbol{\sigma}(\mathbf{u}, p):\mathbf{n}=\mathbf{0}, &\quad \text{at the outflow}, \quad (\kappa(T)\nabla T)\cdot\mathbf{n}=0, \quad \text{at the outflow} \end{aligned} \tag{18}$$

4. MATHEMATICAL ANALYSIS OF THE CONTINUOUS PROBLEM

We shall formulate and analyse a weak form of the steady continuous problem deduced in the previous section. This may seem to be non-sensed *a priori*, because the coupled problem is essentially non-steady in their nature (see [25]), but it has a mathematical interest because the time discretization of the evolution problem by means of backward finite difference formulas leads, in each time step, to a variant of this steady problem (see also the Remark 4 at the end of this section).

From now on, we do not write the indexes 'm' for the Reynolds, Grashof and Péclet numbers (Re_m, Gr_m, Pe_m) but it is important to remember that in their definition there is the reference temperature value T_m chosen at the beginning. Thus, we are interested in solving

$$\mathbf{u}\cdot\nabla\mathbf{u}-\frac{1}{Re}\nabla\cdot(\mu(T)D(\mathbf{u}))+\nabla p=\frac{Gr}{Re^2}T\mathbf{k}+\mathbf{F} \tag{19}$$

$$\nabla\cdot\mathbf{u}=0 \tag{20}$$

$$\mathbf{u}\cdot\nabla T-\frac{1}{Pe}\nabla\cdot(\kappa(T)\nabla T)=0 \tag{21}$$

Let $\Omega\subseteq\mathbb{R}^2$ an open, bounded, convex domain with a Lipschitz boundary $\partial\Omega$. We assume that we have a part $\Gamma_D\subseteq\partial\Omega$ with positive measure, and we set $\Gamma_N=\partial\Omega\backslash\Gamma_D$. The portion Γ_D is associated with the Dirichlet BCs for velocity and temperature and Γ_N is associated with the outflow BCs. Unless other specification, these hypotheses concerning the geometry will be considered hereafter.

We provide Equations (19)–(21) with the following set of BCs:

$$\mathbf{u} = \mathbf{u}_\mathrm{D}, \quad T = T_\mathrm{D} \quad \text{on } \Gamma_\mathrm{D} \tag{22}$$

$$\boldsymbol{\sigma}(\mathbf{u}, p) : \mathbf{n} = \mathbf{0}, \quad (\kappa(T)\nabla T)\cdot\mathbf{n} = 0 \quad \text{on } \Gamma_\mathrm{N} \tag{23}$$

We recall that by definition $\boldsymbol{\sigma}(\mathbf{u}, p) = -p\boldsymbol{\delta} + (1/Re_m)\mu(T)D(\mathbf{u})$. The surface function T_D which takes the prescribed values of temperature at the entry and the walls is supposed uniformly bounded: there exists two real constants T_1 and T_2 such that

$$T_1 \leqslant T_\mathrm{D}(\mathbf{x}) \leqslant T_2 \quad \text{on } \Gamma_\mathrm{D} \tag{24}$$

In virtue of the non-dimensional temperature chosen in the model, we note that T_1 is not necessarily positive. We will show that, for the solution of the coupled problem, and under the assumptions stated in the deduction of the model, the temperature T is bounded by these uniform constants too.

4.1. Weak formulation of the coupled non-linear problem

The set of BCs considered suggests naturally the functional spaces for this study. Let $L^2(\Omega)$ the standard Lebesgue space, with norm $\|\cdot\|_0$. We introduce the following well-known functional vector spaces:

$$H^1(\Omega) = \{v \in L^2(\Omega), \nabla v \in \mathbf{L}^2(\Omega)\} \tag{25}$$

$$H^1_{0,\Gamma_\mathrm{D}} = \{v \in H^1(\Omega), v|_{\Gamma_\mathrm{D}} = 0\} \tag{26}$$

$$H^{1/2}(\Gamma_\mathrm{D}) = \{\eta \in L^2(\Gamma_\mathrm{D}), \exists v \in H^1(\Omega),\ v|_{\Gamma_\mathrm{D}} = \eta\} \tag{27}$$

where $v|_{\Gamma_\mathrm{D}}$ is the partial trace on Γ_D of a function $v \in H^1(\Omega)$. The analogous vector-valued spaces will be denoted by bold symbols, so for example: $\mathbf{H}^1_{0,\Gamma_\mathrm{D}} = H^1_{0,\Gamma_\mathrm{D}} \times H^1_{0,\Gamma_\mathrm{D}}$. The norms for the Hilbert spaces $H^m(\Omega)$ will be denoted by $\|\cdot\|_{m,\Omega}$. The scalar product associated with the $L^2(\Omega)$ norm will be noted $(\cdot,\cdot)_{0,\Omega}$ or just $(\cdot,\cdot)$ if it is unambiguous. In this paper, C, with or without subscript, denotes a generic positive constant depending only on Ω and Γ_D. The value of C may differ at different occurrences.

Following [31], it is well known that for the elements in $H^1_{0,\Gamma_\mathrm{D}}$, we have the Poincaré's inequality and the Korn's inequality

$$\|v\|_{0,\Omega} \leqslant C\|\nabla v\|_{0,\Omega} \quad \forall v \in H^1_{0,\Gamma_\mathrm{D}} \tag{28}$$

$$\|v\|^2_{1,\Omega} \leqslant C(\|v\|^2_{0,\Omega} + \|D(v)\|^2_{0,\Omega}) \quad \forall v \in H^1_{0,\Gamma_\mathrm{D}} \tag{29}$$

The weak form of (19)–(21) is the following: let be $\mathbf{u}_\mathrm{D}$ given in $\mathbf{H}^{1/2}(\Gamma_\mathrm{D})$ and T_D given in $H^{1/2}(\Gamma_\mathrm{D})$. *Find* $(\mathbf{u}, p, T)$ *in* $\mathbf{H}^1(\Omega)\times L^2(\Omega)\times H^1(\Omega)$ *such that* $\mathbf{u}|_{\Gamma_\mathrm{D}} = \mathbf{u}_\mathrm{D}$ *and* $T|_{\Gamma_\mathrm{D}} = T_\mathrm{D}$ *solution of*

$$\mathbf{a}_T(\mathbf{u}, \mathbf{v}) + \mathbf{b}(\mathbf{u}, \mathbf{u}, \mathbf{v}) - (\operatorname{div}\mathbf{v}, p)_{0,\Omega} = \left(\frac{Gr}{Re^2}T\mathbf{k}, \mathbf{v}\right)_{0,\Omega} + (\mathbf{F}, \mathbf{v})_{0,\Omega} \quad \forall \mathbf{v} \in \mathbf{H}^1_{0,\Gamma_\mathrm{D}}(\Omega) \tag{30}$$

$$(\operatorname{div}\mathbf{u}, q)_{0,\Omega} = 0 \quad \forall q \in L^2(\Omega) \tag{31}$$

$$a_T(T, \varphi) + b(\mathbf{u}, T, \varphi) = 0 \quad \forall \varphi \in H^1_{0,\Gamma_\mathrm{D}}(\Omega) \tag{32}$$

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where

$$\mathbf{a}_T(\mathbf{u},\mathbf{v}) = \int_\Omega \frac{1}{Re}\mu(T)D(\mathbf{u}) : \nabla\mathbf{v} \tag{33}$$

$$\mathbf{b}(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w} \tag{34}$$

$$a_T(\psi,\varphi) = \int_\Omega \frac{1}{Pe}\kappa(T)\nabla\psi\cdot\nabla\varphi \tag{35}$$

$$b(\mathbf{u},\psi,\varphi) = \int_\Omega (\mathbf{u}\cdot\nabla)\psi\varphi \tag{36}$$

The notations in Equations (30)–(36) are inspired on the early work of Duvaut and Lions [12], who analysed the viscosity coupling problem for a Bingham fluid. The index for the applications $\mathbf{a}_T$ and a_T have for mission to show the temperature field which is associated with the functions of viscosity and conductivity. We note in particular that the expression $a_T(T,\varphi)$ in Equation (35) makes this equation non-linear.

Thanks to the symmetry of the deformation tensor $D(\mathbf{v})$, the bilinear form $\mathbf{a}_T(\cdot,\cdot)$, continuous on $\mathbf{H}^1_{0,\Gamma_\mathrm{D}}\times\mathbf{H}^1_{0,\Gamma_\mathrm{D}}$ is symmetric since we have

$$\mathbf{a}_T(\mathbf{u},\mathbf{v}) = \int_\Omega \frac{1}{Re}\mu(T)D(\mathbf{u}) : D(\mathbf{v}) \tag{37}$$

From (37), we have naturally that

$$\mathbf{a}_T(\mathbf{u},\mathbf{v})\leqslant\frac{\mu_2 C}{Re}\|\nabla\mathbf{u}\|_{0,\Omega}\|\nabla\mathbf{v}\|_{0,\Omega},\quad \mathbf{a}_T(\mathbf{v},\mathbf{v})\geqslant\frac{\mu_1 C}{Re}\|\nabla\mathbf{v}\|^2_{0,\Omega}$$

with C being the constant given by Korn's inequality (29).

In $\mathbb{R}^2$, the term $\mathbf{b}(\mathbf{u},\mathbf{u},\mathbf{v})$ is defined for $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{H}^1(\Omega)$ since then $\mathbf{u}$ and $\mathbf{v}$ belong to $\mathbf{L}^4(\Omega)$ in virtue of the Sobolev's inclusions in this dimension. The term $b(\mathbf{u},T,\varphi)$ is defined for $\mathbf{u}$ in $\mathbf{H}^1(\Omega)$, T and φ in $H^1(\Omega)$ for the same reasons.

For the convective terms, we recall some well-known identities

$$\mathbf{b}(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{1}{2}\mathbf{b}(\mathbf{u},\mathbf{v},\mathbf{w}) - \frac{1}{2}\mathbf{b}(\mathbf{u},\mathbf{w},\mathbf{v}) + \frac{1}{2}\int_{\partial\Omega}\mathbf{u}\cdot\mathbf{n}\mathbf{v}\cdot\mathbf{w} - \frac{1}{2}\int_\Omega \operatorname{div}\mathbf{u}\,\mathbf{v}\cdot\mathbf{w} \tag{38}$$

$$b(\mathbf{u},T,\varphi) = \frac{1}{2}b(\mathbf{u},T,\varphi) - \frac{1}{2}b(\mathbf{u},\varphi,T) + \frac{1}{2}\int_{\partial\Omega}\mathbf{u}\cdot\mathbf{n}T\varphi - \frac{1}{2}\int_\Omega \operatorname{div}\mathbf{u}\,T\varphi \tag{39}$$

Next, we note that, for $\mathbf{u}$ satisfying $\operatorname{div}\mathbf{u}=0$ in Ω, $\mathbf{v}=0$ on Γ_D and $\varphi=0$ on Γ_D, we obtain

$$\mathbf{b}(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{1}{2}\mathbf{b}(\mathbf{u},\mathbf{v},\mathbf{w}) - \frac{1}{2}\mathbf{b}(\mathbf{u},\mathbf{w},\mathbf{v}) + \frac{1}{2}\int_{\Gamma_\mathrm{N}}\mathbf{u}\cdot\mathbf{n}\,\mathbf{v}\cdot\mathbf{w} \tag{40}$$

$$b(\mathbf{u},T,\varphi) = \frac{1}{2}b(\mathbf{u},T,\varphi) - \frac{1}{2}b(\mathbf{u},\varphi,T) + \frac{1}{2}\int_{\Gamma_\mathrm{N}}\mathbf{u}\cdot\mathbf{n}\,T\varphi \tag{41}$$

As can be expected, without any consideration on the outflow region Γ_{N}, there is no easy way to obtain an *a priori* control of the convective terms (40) and (32) in the weak formulations (30) and (32), because they consider the unknown velocity field $\mathbf{u}$ at this portion of the boundary. This fact was first noted by Heywood *et al*. (see [10]) which also mention their impossibility to obtain an *a priori* estimation similar to the well-known Leray and Hopf techniques to bound the non-homogeneous *steady* problem. They can prove, however, the equivalences between this kind of 'do nothing' BCs (following the notation of Gresho and Sani [32] and Heywood *et al*. [10]) and the physical problem which consider pressure drops and/or prescribed velocity fluxes on the outflow region, by showing how some hidden or implicit BCs appear in the different variational formulations of the given problem.

As our interest in this first part is the analysis of the continuous problem, we shall propose a variational formulation 'closer enough' to the original problem (19)–(23). For this, we follow the ideas proposed for the incompressible isothermal Navier–Stokes equations by Bruneau and Fabrie [11]. It consists on the use of some alternative convective terms which take into account the behaviour of the velocity at the outflow region. Thus, we introduce the forms

$$\tilde{\mathbf{b}}(\mathbf{u},\mathbf{v},\mathbf{w}) = \mathbf{b}(\mathbf{u},\mathbf{v},\mathbf{w}) + \frac{1}{2}\int_{\Gamma_{\mathrm{N}}} [\mathbf{u}\cdot\mathbf{n}]^{-}\mathbf{v}\cdot\mathbf{w} \tag{42}$$

$$\tilde{b}(\mathbf{u},\phi,\varphi) = b(\mathbf{u},\phi,\varphi) + \frac{1}{2}\int_{\Gamma_{\mathrm{N}}} [\mathbf{u}\cdot\mathbf{n}]^{-}\phi\varphi \tag{43}$$

In (42) and (43), $\mathbf{n}$ refers to the unit outward vector to Γ_{N} and $[\cdot]^{-}$ refers to the 'negative part' function, defined by $[f]^{-}(x) = \sup\{-f(x), 0\}$. We also define the 'positive part' function as $[f]^{+}(x) = \sup\{0, f(x)\}$ (see Proposition 4.2).

When (43) is applied to a test function $\varphi \in H^1_{0,\Gamma_{\mathrm{D}}}$ we obtain

$$\tilde{b}(\mathbf{u},\varphi,\varphi) = b(\mathbf{u},\phi,\varphi) + \frac{1}{2}\int_{\Gamma_{\mathrm{N}}} [\mathbf{u}\cdot\mathbf{n}]^{-}\varphi^2 = \frac{1}{2}\int_{\Gamma_{\mathrm{N}}} [\mathbf{u}\cdot\mathbf{n}]^{+}\varphi^2 \geqslant 0 \tag{44}$$

The last value is obtained directly from (41). A similar result is obtained for the evaluation $\tilde{\mathbf{b}}(\mathbf{u},\mathbf{v},\mathbf{v})$ for $\mathbf{v}\in\mathbf{H}^1_{0,\Gamma_{\mathrm{D}}}$. Thus, *a priori* estimates of the homogeneous test functions are available, remaining only the estimates of the non-homogeneous part of the velocity and temperature unknowns.

So, we shall focus on the analysis of the following model: let be $\mathbf{u}_{\mathrm{D}}$ given in $\mathbf{H}^{1/2}(\Gamma_{\mathrm{D}})$ and T_{D} given in $H^{1/2}(\Gamma_{\mathrm{D}})$. *Find* $(\mathbf{u}, p, T)$ *in* $\mathbf{H}^1(\Omega)\times L^2(\Omega)\times H^1(\Omega)$ *such that* $\mathbf{u}|_{\Gamma_{\mathrm{D}}} = \mathbf{u}_{\mathrm{D}}$ *and* $T|_{\Gamma_{\mathrm{D}}} = T_{\mathrm{D}}$ *solution of*

$$\mathbf{a}_T(\mathbf{u},\mathbf{v}) + \tilde{\mathbf{b}}(\mathbf{u},\mathbf{u},\mathbf{v}) - (\operatorname{div}\mathbf{v}, p) = \left(\frac{Gr}{Re^2}T\mathbf{k} + \mathbf{F}, \mathbf{v}\right) \quad \forall \mathbf{v}\in\mathbf{H}^1_{0,\Gamma_{\mathrm{D}}}(\Omega) \tag{45}$$

$$(\operatorname{div}\mathbf{u}, q) = 0 \quad \forall q \in L^2(\Omega) \tag{46}$$

$$a_T(T,\varphi) + \tilde{b}(\mathbf{u}, T, \varphi) = 0 \quad \forall \varphi \in H^1_{0,\Gamma_{\mathrm{D}}}(\Omega) \tag{47}$$

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A solution $(\mathbf{u}, p, T)$ of this weak formulation (45)–(47) is, at least formally, a solution of

$$-\frac{1}{Re}\nabla\cdot(\mu(T)D(\mathbf{u}))+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=\frac{Gr}{Re^2}T\mathbf{k}+\mathbf{F} \tag{48}$$

$$\operatorname{div}\mathbf{u}=0 \tag{49}$$

$$-\frac{1}{Pe}\nabla\cdot(\kappa(T)\nabla T)+\mathbf{u}\cdot\nabla T=0 \tag{50}$$

$$\mathbf{u}=\mathbf{u}_{\mathrm{D}} \text{ on } \Gamma_{\mathrm{D}}, \quad \boldsymbol{\sigma}(\mathbf{u}, p):\mathbf{n}+\tfrac{1}{2}[\mathbf{u}\cdot\mathbf{n}]^{-}\mathbf{u}=0 \text{ on } \Gamma_{\mathrm{N}} \tag{51}$$

$$T=T_{\mathrm{D}} \text{ on } \Gamma_{\mathrm{D}}, \quad \kappa(T)\nabla T\cdot\mathbf{n}+\tfrac{1}{2}[\mathbf{u}\cdot\mathbf{n}]^{-}T=0 \text{ on } \Gamma_{\mathrm{N}} \tag{52}$$

We note that if $(\mathbf{u}, p, T)\in\mathbf{H}^1(\Omega)\times L^2(\Omega)\times H^1(\Omega)$ with $\mathbf{u}=\mathbf{u}_{\mathrm{D}}$, $T=T_{\mathrm{D}}$ on Γ_{D} is a solution of the variational formulation (48)–(52) and is such that $\mathbf{u}\cdot\mathbf{n}\geqslant 0$ on Γ_{N}, obviously it will be also solution of the variational problem (30)–(32). This is an important issue, because a solution $(\mathbf{u}, p, T)$ of the weak formulation (30)–(32) is, at least formally, a solution of the problem (19)–(21). In the case that $\mathbf{u}\cdot\mathbf{n}\leqslant 0$ (re-entrant flow case), then both problems (48)–(52) and (30)–(32) are no longer equivalent, and, up to the author's knowledge, it remains an open problem.

The coupled problem (45)–(47) is strongly non-linear. The strategy chosen for the resolution consists of performing an outer (global) Picard iteration in the thermophysical properties and the buoyancy term. That gives, at each step of an iteration, two decoupled problems, namely:

- We choose a reference temperature $\hat{T}$ and solve the Navier–Stokes problem using $\hat{T}$ in the definition of μ and the buoyancy term (see Equations (60)–(61)).
- We fix the velocity $\mathbf{u}$ and the reference temperature $\hat{T}$ and solve (53) for T.

Another possibility is to perform the global iteration in the velocity, but in this way the energy equation becomes non-linear, and a uniqueness result is necessary in this equation in order to define the complete outer iteration. This second strategy is analysed in [25].

In the following subsections, we will analyse each subproblem generated by considering an outer decoupling with a temperature field $\hat{T}$ and we shall prove that the following application allows a fixed point:

$$\hat{T}\mapsto(\mathbf{u}(\hat{T}), p(\hat{T}))\mapsto T(\mathbf{u}(\hat{T}))$$

We first analyse the linearized (in temperature) energy and Navier–Stokes equations.

4.2. The decoupled energy equation

In this subsection, let $\hat{T}$ the element which makes the global decoupling in temperature for the coupled problem, and let $\mathbf{u}$ a 'frozen' divergence-free field (it will be a solution of the uncoupled Navier–Stokes equations).

If we are interested in the weak problem (47), we must solve the linear problem: *find* $T\in H^1(\Omega)$ *such that* $T|_{\Gamma_{\mathrm{D}}}=T_{\mathrm{D}}$, *solution of*

$$a_{\hat{T}}(T,\varphi)+\tilde{b}(\mathbf{u}, T,\varphi)=0 \quad \forall\varphi\in H^1_{0,\Gamma_{\mathrm{D}}}(\Omega) \tag{53}$$

We note a subtle mathematical consideration: we do not know *a priori* if the temperature solutions will be bounded or not (this will be a consequence of maximum principles). In consequence, at the

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first time, we should extend the domain of definition of conductivity and viscosity functions μ and κ to newer functions $\tilde{\mu}$ and $\tilde{\kappa}$, even if these new functions do not have a physical sense. In order to keep the natural properties of the original function, the following extension of the functions is classical:

$$\tilde{\mu}(T)=\begin{cases}\mu(T_1) & \text{if } T<T_1,\\ \mu(T) & \text{if } T_1\leqslant T\leqslant T_2,\\ \mu(T_2) & \text{if } T>T_2,\end{cases}\qquad \tilde{\kappa}(T)=\begin{cases}\kappa(T_1) & \text{if } T<T_1\\ \kappa(T) & \text{if } T_1\leqslant T\leqslant T_2\\ \kappa(T_2) & \text{if } T>T_2\end{cases} \tag{54}$$

That is, we consider constant-bounded extensions of the domain of definition, keeping the uniform bounds given in (17).

It is also convenient in the following to work with the 'homogeneous part' of the temperature. For that, let $T^*\in H^1(\Omega)$ be a lifting of T_D: $T^*|_{\Gamma_\mathrm{D}}=T_\mathrm{D}$. As in [17], we can assume that the lifting is continuous from $H^{1/2}(\Gamma_\mathrm{D})$ into $H^1(\Omega)$ and hence

$$\|T^*\|_{1,\Omega}\leqslant C\|T_\mathrm{D}\|_{1/2,\partial\Omega} \tag{55}$$

With this lifting, we can formulate the equivalent of (53) in terms of the homogeneous part $\theta=T-T^*\in H^1_{0,\Gamma_\mathrm{D}}(\Omega)$. Under the geometry assumptions considered at the beginning of this section, the following proposition resumes the existence and uniqueness result for this problem.

Proposition 4.1
Let $\mathbf{u}\in\mathbf{H}^1(\Omega)$ be a given divergence-free velocity field. Then, the weak problem: find $\theta\in H^1_{0,\Gamma_\mathrm{D}}(\Omega)$ such that

$$a_{\hat{T}}(\theta,\varphi)+\tilde{b}(\mathbf{u},\theta,\varphi)=-a_{\hat{T}}(T^*,\varphi)-\tilde{b}(\mathbf{u},T^*,\varphi)\quad \forall\varphi\in H^1_{0,\Gamma_\mathrm{D}}(\Omega) \tag{56}$$

admits an unique solution in $H^1_{0,\Gamma_\mathrm{D}}$.

Proof
The proof is a direct consequence of Lax–Milgram lemma. For a fixed $\mathbf{u}\in\mathbf{H}^1_{0,\Gamma_\mathrm{D}}$, it is clear that the left term of (56) defines a bilinear form in $H^1_{0,\Gamma_\mathrm{D}}(\Omega)^2$, and the right-hand side defines a linear form in $H^1_{0,\Gamma_\mathrm{D}}(\Omega)$ thanks to Sobolev embeddings and trace theorems.

The ellipticity is obtained this time thanks to (44), since for all $\varphi\in H^1_{0,\Gamma_\mathrm{D}}$

$$b(\mathbf{u},\varphi,\varphi)+\frac{1}{2}\int_{\Gamma_\mathrm{N}}[\mathbf{u}\cdot\mathbf{n}]^-\varphi^2=\frac{1}{2}\int_{\Gamma_\mathrm{N}}[\mathbf{u}\cdot\mathbf{n}]^+\varphi^2\geqslant 0 \tag{57}$$

From (57) the ellipticity condition follows. Moreover, by using the triangular inequality, the bound of the lifting T^* given in (55) and the fact that $\kappa_2/\kappa_1\geqslant 1$, it is easy to obtain the following bound for the temperature T

$$\|T\|_{1,\Omega}\leqslant\tilde{C}\left(\frac{\kappa_2}{\kappa_1}+\frac{Pe}{\kappa_1}\|\mathbf{u}\|_{1,\Omega}\right)\|T_\mathrm{D}\|_{1/2,\Gamma_\mathrm{D}} \tag{58}$$

This proves the existence and uniqueness of θ solution of (56) and then also the existence and uniqueness of T solution of (53). Note that this *a priori* bound of the temperature T depends on the velocity field $\mathbf{u}$. □

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The next step is to show that the solution of (56) is physically allowable in the following sense: because no internal heat generation is considered in the model, the solution must remain bounded by the boundary values. This fact allows us to neglect the extensions built in (54) for the viscosity and conductivity functions and to continue the analysis with the original non-dimensional thermophysical properties.

Proposition 4.2
Let $\mathbf{u}\in\mathbf{H}^1(\Omega)$ be a given divergence-free velocity field, and let us suppose that the Dirichlet data $T_\mathrm{D}\in H^{1/2}(\Gamma_\mathrm{D})$ is such that

$$T_1\leqslant T_\mathrm{D}(x)\leqslant T_2 \quad \text{a.e. on } \Gamma_\mathrm{D}$$

Then, problem (53) has an unique solution T that verifies

$$T_1\leqslant T(x)\leqslant T_2 \quad \text{a.e. in } \Omega$$

Proof
Following the Marcus–Mitzel and Rademacher theorems (see [33, 34]), the function $\psi=[T-T_2]^+$ belongs to $H^1(\Omega)$. Moreover, by construction, it verifies that $\psi=0$ in Γ_D. With this choice of test function ψ in Equation (53), we obtain

$$a_{\hat{T}}(T,[T-T_2]^+)+\tilde{b}(\mathbf{u},T,[T-T_2]^+)=0$$

The rest of the proof consists of analysing the supports of each term $[T-T_2]^+$ in the integrals (here referred as $[T\geqslant T_2]$). In the region where $T-T_2\geqslant 0$, we have $(T-T_2)=[T-T_2]^+$. Also, since T_2 is constant, in the support $[T\geqslant T_2]$ we have $\nabla T=\nabla(T-T_2)=\nabla[T-T_2]^+$. Thus,

$$\int_{\Omega\cap[T\geqslant T_2]}\frac{\tilde{\kappa}(\hat{T})}{Pe}\nabla[T-T_2]^+\cdot\nabla[T-T_2]^+ +\int_{\Omega\cap[T\geqslant T_2]}(\mathbf{u}\cdot\nabla[T-T_2]^+)[T-T_2]^+$$
$$+\frac{1}{2}\int_{\Gamma_\mathrm{N}\cap[T\geqslant T_2]}[\mathbf{u}\cdot\mathbf{n}]^-[T-T_2]^+[T-T_2]^+=0 \tag{59}$$

Next, because of (57), we can neglect the last two terms in (59) and obtain:

$$\frac{\kappa_1}{Pe}\int_{\Omega\cap[T\geqslant T_2]}|\nabla[T-T_2]^+|^2\leqslant 0$$

Since $[T-T_2]^+\in H^1_{0,\Gamma_\mathrm{D}}(\Omega)$, we conclude that

$$[T-T_2]^+=0$$

which means

$$T\leqslant T_2 \quad \text{a.e. on } \Omega$$

By choosing as test function $\psi=-[T-T_1]^-$, we obtain likewise

$$T\geqslant T_1 \quad \text{a.e. on } \Omega$$

and the result follows. □

Remark 1
In virtue of Proposition 4.2, with a datum T_D verifying $T_1 \leqslant T_D \leqslant T_2$ on Γ_D, from now on we can omit the extensions $\tilde{\kappa}$ and $\tilde{\mu}$. Thus, we continue with the original viscosity and conductivity functions μ and κ.

We now analyse the uncoupled Navier–Stokes equation.

4.3. The decoupled Navier–Stokes equations

Let $\hat{T} \in H^1(\Omega)$ be fixed in this section; we shall analyse the following weak Navier–Stokes problem: *find* $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)$ *with* $\mathbf{u} = \mathbf{u}_D$ *on* Γ_D, *solution of*

$$\mathbf{a}_{\hat{T}}(\mathbf{u}, \mathbf{v}) + \tilde{\mathbf{b}}(\mathbf{u}, \mathbf{u}, \mathbf{v}) - (\operatorname{div} \mathbf{v}, p) = \left(\frac{Gr}{Re^2} \hat{T} \mathbf{k} + \mathbf{F}, \mathbf{v} \right) \quad \forall \mathbf{v} \in \mathbf{H}^1_{0,\Gamma_D}(\Omega) \tag{60}$$

$$-(\operatorname{div} \mathbf{u}, q) = 0 \quad \forall q \in L^2(\Omega) \tag{61}$$

We note in (60)–(61) that the linearization is realized globally for the temperature field, so at this level, there is a 'frozen' temperature field $\hat{T}$ in the viscous and the buoyant term, but the Navier–Stokes equations remain non-linear in velocity. In addition, in order to define a global iteration (temperature→velocity→temperature) it is necessary to state not only an existence, but also an uniqueness result for the velocity solutions. This is the aim of the following proposition.

Proposition 4.3
For any $\mathbf{F}$ given in $L^2(\Omega)$, for any $\hat{T}$ given in $L^2(\Omega)$, with $T_1 \leqslant \hat{T} \leqslant T_2$ and for any $\mathbf{u}_D$ given in $\mathbf{H}^{1/2}(\Gamma_D)$, the problem: *find* $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega) \times L^2(\Omega)$ *with* $\mathbf{u} = \mathbf{u}_D$ *on* Γ_D, *solution of* (60)–(61) has at least one solution.

Proof
As for the previous section, we proceed to formulate the problem in the 'homogeneous' part of the velocity field: $\mathbf{w} = \mathbf{u} - \mathbf{u}^*$, where $\mathbf{u}^*$ is a convenient lifting of the Dirichlet BCs. Thus, to the Dirichlet data $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma_D)$, we associate a function $\mathbf{u}^* \in \mathbf{H}^1(\Omega)$ such that

$$\mathbf{u}^* = \mathbf{u}_D \ \text{ on } \Gamma_D \quad \text{and} \quad \operatorname{div} \mathbf{u}^* = 0 \ \text{ in } \Omega$$

The choice of $\mathbf{u}^*$ will be precised later for satisfying a specific bound. Once this function $\mathbf{u}^*$ is selected, it remains fixed for the rest of the analysis.

Once this lifting is chosen, the Navier–Stokes problem (60)–(61) is written as: *find* $(\mathbf{w}, p) \in \mathbf{H}^1_{0,\Gamma_D}(\Omega) \times L^2(\Omega)$ *solution of*

$$\mathbf{a}_{\hat{T}}(\mathbf{w} + \mathbf{u}^*, \mathbf{v}) + \tilde{\mathbf{b}}(\mathbf{w} + \mathbf{u}^*, \mathbf{w} + \mathbf{u}^*, \mathbf{v}) - (\operatorname{div} \mathbf{v}, p) = \left(\frac{Gr}{Re^2} \hat{T} \mathbf{k} + \mathbf{F}, \mathbf{v} \right) \quad \forall \mathbf{v} \in \mathbf{H}^1_{0,\Gamma_D}(\Omega) \tag{62}$$

$$-(\operatorname{div} \mathbf{w}, q) = 0 \quad \forall q \in L^2(\Omega) \tag{63}$$

Let us introduce the forms

$$\widetilde{\mathbf{a}}(\mathbf{v}_0; \mathbf{v}_1, \mathbf{v}_2) = \widetilde{\mathbf{a}}_0(\mathbf{v}_1, \mathbf{v}_2) + \widetilde{\mathbf{a}}_1(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2) \tag{64}$$

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with

$$\widetilde{\mathbf{a}}_0(\mathbf{v}_1,\mathbf{v}_2)=\mathbf{a}_{\hat{T}}(\mathbf{v}_1,\mathbf{v}_2)+\mathbf{b}(\mathbf{v}_1,\mathbf{u}^*,\mathbf{v}_2)+\frac{1}{2}\int_{\Gamma_{\mathrm{N}}}([(\mathbf{v}_1+\mathbf{u}^*)\cdot\mathbf{n}]^- - [\mathbf{u}^*\cdot\mathbf{n}]^-)\mathbf{u}^*\cdot\mathbf{v}_2 \tag{65}$$

$$\widetilde{\mathbf{a}}_1(\mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_2)=\mathbf{b}(\mathbf{v}_0+\mathbf{u}^*,\mathbf{v}_1,\mathbf{v}_2)+\frac{1}{2}\int_{\Gamma_{\mathrm{N}}}[(\mathbf{v}_0+\mathbf{u}^*)\cdot\mathbf{n}]^-\mathbf{v}_1\cdot\mathbf{v}_2 \tag{66}$$

and

$$L(\mathbf{v})=\left(\frac{Gr}{Re^2}\hat{T}\mathbf{k}+\mathbf{F},\mathbf{v}\right)-\tilde{\mathbf{b}}(\mathbf{u}^*,\mathbf{u}^*,\mathbf{v}) \tag{67}$$

With these notations, we write (62)–(63) as: *find* $(\mathbf{w},p)\in\mathbf{H}^1_{0,\Gamma_{\mathrm{D}}}(\Omega)\times L^2(\Omega)$ *solution of*

$$\widetilde{\mathbf{a}}(\mathbf{w};\mathbf{w},\mathbf{v})-(\operatorname{div}\mathbf{v},p)=L(\mathbf{v})\quad\forall\mathbf{v}\in\mathbf{H}^1_{0,\Gamma_{\mathrm{D}}}(\Omega) \tag{68}$$

$$-(\operatorname{div}\mathbf{w},q)=0\quad\forall q\in L^2(\Omega) \tag{69}$$

On the one hand, according to the compactness of the embedding of $\mathbf{H}^1_{0,\Gamma_{\mathrm{D}}}(\Omega)$ into $\mathbf{L}^4(\Omega)$, to the compactness of the trace mapping from $\mathbf{H}^1_{0,\Gamma_{\mathrm{D}}}(\Omega)$ into $\mathbf{L}^3(\Gamma_{\mathrm{N}})$, for each $\mathbf{v}_2\in\mathbf{H}^1_{0,\Gamma_{\mathrm{D}}}(\Omega)$ the mapping $\mathbf{v}_1\mapsto\widetilde{\mathbf{a}}(\mathbf{v}_1;\mathbf{v}_1,\mathbf{v}_2)$ is weakly continuous on $\mathbf{H}^1_{0,\Gamma_{\mathrm{D}}}(\Omega)$. On the other hand, we will prove, see below, that with a convenient choice of the lifting $\mathbf{u}^*$, the form $\widetilde{\mathbf{a}}$ is elliptic in the sense that there exists a positive number α, which may be selected independent of the physical constants related to the fluid and independent of $\hat{T}$, such that

$$\forall\mathbf{v}\in\mathbf{H}^1_{0,\Gamma_{\mathrm{D}}}(\Omega)\ \text{with}\ \operatorname{div}\mathbf{v}=0\ \text{in}\ \Omega,\quad \widetilde{\mathbf{a}}(\mathbf{v};\mathbf{v},\mathbf{v})\geqslant\alpha\frac{\mu_1}{Re}\|\mathbf{v}\|^2_{1,\Omega} \tag{70}$$

Then, the problem: find $\mathbf{w}\in\mathbf{H}^1_{0,\Gamma_{\mathrm{D}}}(\Omega)$ with $\operatorname{div}\mathbf{w}=0$ in Ω, solution of

$$\widetilde{\mathbf{a}}(\mathbf{w};\mathbf{w},\mathbf{v})=L(\mathbf{v})\quad\forall\mathbf{v}\in\mathbf{H}^1_{0,\Gamma_{\mathrm{D}}}(\Omega)\ \text{with}\ \operatorname{div}\mathbf{v}=0\ \text{in}\ \Omega \tag{71}$$

has at least one solution $\mathbf{w}\in\mathbf{H}^1_{0,\Gamma_{\mathrm{D}}}(\Omega)$ (apply, for example, the Theorem IV.2.1 in [3]). To a solution of this problem, we associate $\mathbf{u}=\mathbf{w}+\mathbf{u}^*$ and we consider the problem: find $p\in L^2(\Omega)$ solution of

$$(\operatorname{div}\mathbf{v},p)=l(\mathbf{v})\quad\forall\mathbf{v}\in\mathbf{H}^1_{0,\Gamma_{\mathrm{D}}}(\Omega) \tag{72}$$

with

$$l(\mathbf{v})=\mathbf{a}_{\hat{T}}(\mathbf{u},\mathbf{v})+\tilde{\mathbf{b}}(\mathbf{u},\mathbf{u},\mathbf{v})-\left(\frac{Gr}{Re^2}\hat{T}\mathbf{k}+\mathbf{F},\mathbf{v}\right) \tag{73}$$

The linear form $\mathbf{v}\mapsto l(\mathbf{v})$ is continuous on $\mathbf{H}^1_{0,\Gamma_{\mathrm{D}}}(\Omega)$ and the bilinear form $(\mathbf{v},q)\mapsto(\operatorname{div}\mathbf{v},q)$ is continuous on $\mathbf{H}^1_{0,\Gamma_{\mathrm{D}}}(\Omega)\times L^2(\Omega)$. Moreover, for any $q\in L^2(\Omega)$ we can construct $\mathbf{g}\in\mathbf{H}^{1/2}(\partial\Omega)$ such that $\mathbf{g}=0$ on Γ_{D}, $\int_{\partial\Omega}\mathbf{g}\cdot\mathbf{n}=\int_{\Gamma_{\mathrm{N}}}\mathbf{g}\cdot\mathbf{n}=\int_{\Omega}q$ and $\|\mathbf{g}\|_{1/2,\partial\Omega}\leqslant C\|q\|_{0,\Omega}$. With such a function $\mathbf{g}$, we can find $\mathbf{v}\in\mathbf{H}^1(\Omega)$ such that $\mathbf{v}=\mathbf{g}$ on $\partial\Omega$, $\operatorname{div}\mathbf{v}=q$ on Ω and $\|\mathbf{v}\|_{1,\Omega}\leqslant C\|\mathbf{g}\|_{1/2,\partial\Omega}$

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(see [35]). That proves the inf–sup condition

$$\inf_{q \in L^2(\Omega)} \sup_{\mathbf{v} \in \mathbf{H}^1_{0,\Gamma_D}(\Omega)} \frac{(\operatorname{div} \mathbf{v}, q)}{\|\mathbf{v}\|_{1,\Omega} \|q\|_{0,\Omega}} \geqslant \beta \tag{74}$$

for some positive number β. We conclude the existence and uniqueness (for $\mathbf{u}$ fixed) of $p \in L^2(\Omega)$ solution of (72) by the Babuška theory (see [36]) and evidently the pair $(\mathbf{u}, p)$ is solution of (60)–(61). So, the proof of the Proposition 4.3 will be complete when the ellipticity of $\widetilde{\mathbf{a}}$ referred in (70) is proved.

Proof of ellipticity (70) of $\widetilde{\mathbf{a}}$*, given by (64)–(66)*
For all divergence-free $\mathbf{v} \in \mathbf{H}^1_{0,\Gamma_D}(\Omega)$, with (44) we have

$$\begin{aligned} \widetilde{\mathbf{a}}_1(\mathbf{v}; \mathbf{v}, \mathbf{v}) &= \tilde{\mathbf{b}}(\mathbf{v} + \mathbf{u}^*, \mathbf{v}, \mathbf{v}) \\ &= \frac{1}{2} \int_{\Gamma_N} [(\mathbf{v} + \mathbf{u}^*) \cdot \mathbf{n}]^+ \mathbf{v} \cdot \mathbf{v} \end{aligned}$$

which leads to

$$\widetilde{\mathbf{a}}_1(\mathbf{v}; \mathbf{v}, \mathbf{v}) \geqslant 0 \tag{75}$$

The Körn and Poincaré's inequalities lead to the existence of a positive number α, which depends only on the geometry of Ω and the decomposition of the boundary into Γ_D and Γ_N such that, for all $\mathbf{v} \in \mathbf{H}^1_{0,\Gamma_D}(\Omega)$,

$$\mathbf{a}_{\hat{T}}(\mathbf{v}, \mathbf{v}) \geqslant 2\alpha \frac{\mu_1}{Re} \|\mathbf{v}\|^2_{1,\Omega} \tag{76}$$

Finally, as a generalization of the Hopf's Lemma, for any $\varepsilon > 0$ there exists an element $\mathbf{u}^* \in \mathbf{H}^1(\Omega)$ such that

$$\begin{gathered} \mathbf{u}^* = \mathbf{u}_D \text{ on } \Gamma_D, \quad \operatorname{div} \mathbf{u}^* = 0 \text{ in } \Omega \\ \forall \mathbf{v} \in \mathbf{H}^1_{0,\Gamma_D}(\Omega), \left| \mathbf{b}(\mathbf{v}, \mathbf{u}^*, \mathbf{v}) + \frac{1}{2} \int_{\Gamma_N} ([(\mathbf{v} + \mathbf{u}^*) \cdot \mathbf{n}]^- - [\mathbf{u}^* \cdot \mathbf{n}]^- \mathbf{u}^* \cdot \mathbf{v} \right| \leqslant \varepsilon \|\mathbf{v}\|^2_{1,\Omega} \end{gathered} \tag{77}$$

(Indication for proving this technical result: generalize the construction given in [3, pp. 287–291], with a cut-off function $\theta_\varepsilon = 1$ in a neighbourhood of Γ_D and $\theta_\varepsilon(x) = 0$ if $d(x, \Gamma_D) \leqslant 2\exp(-1/\varepsilon)$. More details in [37].)

The proof also uses the following bound:

$$\|[(\mathbf{v} + \mathbf{u}^*) \cdot \mathbf{n}]^- - [\mathbf{u}^* \cdot \mathbf{n}]^-\|_{L^3(\Gamma_N)} \leqslant \|\mathbf{v} \cdot \mathbf{n}\|_{L^3(\Gamma_N)} \leqslant C \|\mathbf{v}\|_{1,\Omega} \tag{78}$$

From (75) to (77), we deduce for all $\varepsilon > 0$

$$\forall \mathbf{v} \in \mathbf{H}^1_{0,\Gamma_D}(\Omega) \text{ with } \operatorname{div} \mathbf{v} = 0 \text{ on } \Omega, \quad \widetilde{\mathbf{a}}(\mathbf{v}; \mathbf{v}, \mathbf{v}) \geqslant \left(2\alpha \frac{\mu_1}{Re} - \varepsilon\right) \|\mathbf{v}\|^2_{1,\Omega} \tag{79}$$

The choice $\varepsilon = \alpha(\mu_1/Re)$ in (79) furnishes the desired ellipticity property (70). □

We complement the previous proposition with the following result concerning the bounds of the solutions of (60)–(61).

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Proposition 4.4
Let us assume that the hypotheses of Proposition 4.3 hold. Then, there exists a constant C, depending on Ω and Γ_D, such that any solution $(\mathbf{u}, p)$ of the Navier–Stokes problem (60)–(61) satisfies

$$\|\mathbf{u}\|_{1,\Omega} \leqslant C\left(\|\mathbf{u}_D\|_{1/2,\Gamma_D} + \frac{Re}{\mu_1}\|\mathbf{u}_D\|^2_{1/2,\Gamma_D} + \frac{Gr}{\mu_1 Re}\|\hat{T}\|_{0,\Omega} + \frac{Re}{\mu_1}\|\mathbf{F}\|_{0,\Omega}\right) \tag{80}$$

and

$$\|p\|_{0,\Omega} \leqslant C\left(\mu_2\|\mathbf{u}_D\|^2_{1/2,\Gamma_D} + \frac{Gr}{Re^2}\|\hat{T}\|_{0,\Omega} + \|\mathbf{F}\|_{0,\Omega} + \mu_2\|\mathbf{u}\|^2_{1,\Omega}\right) \tag{81}$$

Remark 2
The bounds on (80) and (81) depend on the temperature field which makes the uncoupling.

Proof of the proposition
From (68), if we consider as test function $\mathbf{w} = \mathbf{u} - \mathbf{u}^*$, estimation (70) reads in particular

$$\frac{\mu_1}{Re}\|\mathbf{w}\|_{1,\Omega} \leqslant C\left(\frac{Gr}{Re^2}\|\hat{T}\|_{0,\Omega} + \|\mathbf{F}\|_{0,\Omega} + \|\mathbf{u}^*\|^2_{1,\Omega}\right) \tag{82}$$

Besides, $\|\mathbf{u}^*\|_{1,\Omega} \leqslant C\|\mathbf{u}_D\|_{1/2,\Gamma_D}$, from which we obtain the bound for $\|\mathbf{w}\|_{1,\Omega}$. The triangular inequality gives the desired estimate (80).

For the pressure estimate, again from (68), we have

$$(\operatorname{div} \mathbf{v}, p) = \widetilde{\mathbf{a}}(\mathbf{w}, \mathbf{w}, \mathbf{v}) - L(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^1_{0,\Gamma_D}(\Omega) \tag{83}$$

The use of the inf–sup condition (74) gives

$$\|p\|_{0,\Omega} \leqslant C\left(\frac{Gr}{Re^2}\|\hat{T}\|_{0,\Omega} + \|\mathbf{F}\|_{0,\Omega} + \mu_2\|\mathbf{w}\|^2_{1,\Omega}\right) \tag{84}$$

And the estimate $\|\mathbf{w}\|^2_{1,\Omega} \leqslant C(\|\mathbf{u}\|^2_{1,\Omega} + \|\mathbf{u}_D\|^2_{1/2,\Gamma_D})$ leads to (81). □

We finish the analysis of the uncoupled Navier–Stokes problem (60)–(61) with a necessary uniqueness result, in order to be able to define the global decoupling in temperature. As usual (see Theorem IV.2.2 in [3]), the uniqueness will be possible to state under suitable data.

Proposition 4.5
Let the hypotheses of the Proposition 4.3 be verified. Then, for sufficiently small Reynolds and Péclet numbers, and small values of the prescribed boundary values T_D and $\mathbf{u}_D$, the Navier–Stokes problem (60)–(61) admits an unique solution.

Proof
The proof follows the same strategy than the constant-property case analysed by Lions (cf. [4]). Let $(\mathbf{u}_1, p_1)$ and $(\mathbf{u}_2, p_2)$ be two solutions of the Navier–Stokes problem (60)–(61) belonging to

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$\mathbf{H}^1(\Omega)\times L^2(\Omega)$. Their difference yields for each test function $\mathbf{v}\in\mathbf{H}^1_{0,\Gamma_D}(\Omega)$ and all $q\in L^2(\Omega)$

$$\mathbf{a}_{\hat{T}}(\mathbf{u}_1-\mathbf{u}_2,\mathbf{v})+\tilde{\mathbf{b}}(\mathbf{u}_1,\mathbf{u}_1,\mathbf{v})-\tilde{\mathbf{b}}(\mathbf{u}_2,\mathbf{u}_2,\mathbf{v})-(\operatorname{div}\mathbf{v},p_1-p_2)=0 \tag{85}$$

$$(-\operatorname{div}(\mathbf{u}_1-\mathbf{u}_2),q)=0 \tag{86}$$

The choice $\mathbf{v}=\mathbf{u}_1-\mathbf{u}_2$ belongs to $\mathbf{H}^1_{0,\Gamma_D}(\Omega)$ and hence it can be considered as test function. Their insertion in (85) holding account (86) yields

$$\begin{aligned}&\mathbf{a}_{\hat{T}}(\mathbf{u}_1-\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_2)+\mathbf{b}(\mathbf{u}_1-\mathbf{u}_2,\mathbf{u}_1,\mathbf{u}_1-\mathbf{u}_2)+\mathbf{b}(\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_2)\\&\quad+\frac{1}{2}\int_{\Gamma_N}[\mathbf{u}_1\cdot\mathbf{n}]^-\mathbf{u}_1\cdot(\mathbf{u}_1-\mathbf{u}_2)-\frac{1}{2}\int_{\Gamma_N}[\mathbf{u}_2\cdot\mathbf{n}]^-\mathbf{u}_2\cdot(\mathbf{u}_1-\mathbf{u}_2)=0\end{aligned} \tag{87}$$

and from (87) we have finally

$$\begin{aligned}&\mathbf{a}_{\hat{T}}(\mathbf{u}_1-\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_2)+\mathbf{b}(\mathbf{u}_1-\mathbf{u}_2,\mathbf{u}_1,\mathbf{u}_1-\mathbf{u}_2)+\mathbf{b}(\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_2)\\&\quad+\frac{1}{2}\int_{\Gamma_N}[\mathbf{u}_1\cdot\mathbf{n}]^-(\mathbf{u}_1-\mathbf{u}_2)\cdot(\mathbf{u}_1-\mathbf{u}_2)\\&\quad+\frac{1}{2}\int_{\Gamma_N}([\mathbf{u}_1\cdot\mathbf{n}]^--[\mathbf{u}_2\cdot\mathbf{n}]^-)\,\mathbf{u}_2\cdot(\mathbf{u}_1-\mathbf{u}_2)=0\end{aligned} \tag{88}$$

In (88) we proceed to the majoration: with the ellipticity condition (76), the continuity of the convective terms, the boundness of the H^1-norm of the solutions $\mathbf{u}_1$ and $\mathbf{u}_2$ and the use of bound (78) gives

$$\begin{aligned}2\alpha\frac{\mu_1}{Re}\|\mathbf{u}_1-\mathbf{u}_2\|^2_{1,\Omega}\leqslant{}& C(\|\mathbf{u}_1-\mathbf{u}_2\|^2_{1,\Omega}(\|\mathbf{u}_1\|_{1,\Omega}+\|\mathbf{u}_2\|_{1,\Omega}))\\&+\|\mathbf{u}_1-\mathbf{u}_2\|^2_{\mathbf{L}^3(\Gamma_N)}(\|\mathbf{u}_1\|_{\mathbf{L}^3(\Gamma_N)}+\|\mathbf{u}_2\|_{\mathbf{L}^3(\Gamma_N)})\end{aligned} \tag{89}$$

and thanks to the compact embeddings, we can write

$$2\alpha\frac{\mu_1}{Re}\|\mathbf{u}_1-\mathbf{u}_2\|^2_{1,\Omega}\leqslant C_0(\|\mathbf{u}_1\|_{1,\Omega}+\|\mathbf{u}_2\|_{1,\Omega})\|\mathbf{u}_1-\mathbf{u}_2\|^2_{1,\Omega} \tag{90}$$

By using (80), we can read (90) as

$$\begin{aligned}&2\alpha\frac{\mu_1}{Re}\|\mathbf{u}_1-\mathbf{u}_2\|^2_{1,\Omega}\\&\quad\leqslant 2C\left(\|\mathbf{u}_D\|_{1/2,\Gamma_D}+\frac{Re}{\mu_1}\|\mathbf{u}_D\|^2_{1/2,\Gamma_D}+\frac{Gr}{\mu_1 Re}\|\hat{T}\|_{0,\Omega}+\frac{Re}{\mu_1}\|\mathbf{F}\|_{0,\Omega}\right)\|\mathbf{u}_1-\mathbf{u}_2\|^2_{1,\Omega}\end{aligned} \tag{91}$$

for a constant C which is now the product of the constant C that appears in (80) and the constant C_0 in (90). In consequence, we obtain the uniqueness of the solution of problem (60)–(61) if

$$C\left(\|\mathbf{u}_D\|_{1/2,\Gamma_D}+\frac{Re}{\mu_1}\|\mathbf{u}_D\|^2_{1/2,\Gamma_D}+\frac{Gr}{\mu_1 Re}\|\hat{T}\|_{0,\Omega}+\frac{Re}{\mu_1}\|\mathbf{F}\|_{0,\Omega}\right)<\alpha\frac{\mu_1}{Re} \tag{92}$$

Hereafter, ‘small data’ will mean that estimate (92) is verified. □

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4.4. The coupled problem

In the previous sections, we have analysed the uncoupled subproblems which come from the following model

$$\mathbf{a}_T(\mathbf{u},\mathbf{v})+\tilde{\mathbf{b}}(\mathbf{u},\mathbf{u},\mathbf{v})-(\operatorname{div}\mathbf{v},p)=\left(\frac{Gr}{Re^2}T\mathbf{k}+\mathbf{F},\mathbf{v}\right)\quad\forall\mathbf{v}\text{ in }\mathbf{H}^1_{0,\Gamma_{\mathrm{D}}}(\Omega) \tag{93}$$

$$(\operatorname{div}\mathbf{u},q)=0\quad\forall q\text{ in }L^2(\Omega) \tag{94}$$

$$a_T(T,\varphi)+\tilde{b}(\mathbf{u},T,\varphi)=0\quad\forall\varphi\text{ in }H^1_{0,\Gamma_{\mathrm{D}}}(\Omega) \tag{95}$$

We have proved that for a temperature field realizing the uncoupling, and under the hypotheses of Propositions 4.3 and 4.5, the (linear) energy equation and the Navier–Stokes equations admit a unique solution. Thus, the following iteration is well defined

$$\hat{T}\mapsto(\mathbf{u}(\hat{T}),p(\hat{T}))\mapsto T(\mathbf{u}(\hat{T}))$$

However, in the previous analysis, the uniform bounds for each subproblem depend on the solution of the complementary one and the temperature field which realizes the uncoupling. More precisely, for an arbitrary fixed $\hat{T}$, which fixes the thermophysical properties and makes the uncoupling of problem (93)–(95), the solution T of energy equation (95) verifies the uniform bound

$$\|T\|_{1,\Omega}\leqslant C\left(\frac{\kappa_2}{\kappa_1}+\frac{Pe}{\kappa_1}\|\mathbf{u}\|_{1,\Omega}\right)\|T_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}} \tag{96}$$

Note that this bound does not depend explicitly on $\hat{T}$, but implicitly by $\mathbf{u}=\mathbf{u}(\hat{T})$.

In the same way, let $\hat{T}$, which fixes the thermophysical property of viscosity and buoyancy. The solution $(\mathbf{u},p)$ of the Navier–Stokes problem (60)–(61) verifies the following estimates

$$\|\mathbf{u}\|_{1,\Omega}\leqslant C\left(\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma_{\mathrm{D}}}+\frac{Re}{\mu_1}\|\mathbf{u}_{\mathrm{D}}\|^2_{1/2,\Gamma_{\mathrm{D}}}+\frac{Gr}{\mu_1 Re}\|\hat{T}\|_{0,\Omega}+\frac{Re}{\mu_1}\|\mathbf{F}\|_{0,\Omega}\right) \tag{97}$$

$$\|p\|_{0,\Omega}\leqslant C\left(\mu_2\|\mathbf{u}_{\mathrm{D}}\|^2_{1/2,\Gamma_{\mathrm{D}}}+\frac{Gr}{Re^2}\|\hat{T}\|_{0,\Omega}+\|\mathbf{F}\|_{0,\Omega}+\mu_2\|\mathbf{u}\|^2_{1,\Omega}\right) \tag{98}$$

We note in (97) and (98) that the velocity and pressure depend on the temperature field which realizes the uncoupling.

Remark 3

From estimates (96)–(98), we note that if the Grashof number is zero ($Gr=0$, that is, no buoyant term), we can obtain from (97) an uniform bound for $\mathbf{u}$ independent of the temperature field which makes the uncoupling, and this estimate in (98) gives an uniform bound for the pressure and in (96) a uniform bound for the temperature field, which as a sense if the data of the problem is small enough. In fact, the constant property and non-buoyant problem, which correspond to the situation analysed by Bruneau and Fabrie (see [13]) matches on this case. In a such situation, one

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can define the convex, closed, non-empty, bounded set of solutions $(\mathbf{u}, p, T)$ bounded by these uniform bounds, and use the compact embeddings of $H_0^1(\Gamma_D)$ into $L^2(\Omega)$ in order to prove, with sequentially continuity arguments valid on this closed bounded convex set, that the hypotheses of the Schauder's fixed point theorem apply. The technique in this situation follows immediately from the standard case shown in Gagneux and Madaune-Tort [33]. This remark is stated in order to show that the existence (and uniqueness) when $Gr=0$ follows with the same techniques utilized in the constant-property case.

However, when $Gr>0$, estimates (96) and (98) are not uniform, and the existence results will need additional hypotheses.

First of all, let us introduce some notation. For an arbitrary $\hat{T}$, let us define the following maps:

$$S : \hat{T} \mapsto S(\hat{T})$$

$$\text{Unique solution of Navier–Stokes problem (60)–(61)} \tag{99}$$

$$Z : \hat{T} \mapsto Z_{\hat{T}}$$

$$\text{Linearized energy problem (53)} \tag{100}$$

That is, $S(\hat{T})$ denotes the unique solution of the Navier–Stokes problem (60)–(61) for a given temperature field $\hat{T}$ which defines the viscosity function and the buoyant term. The application Z defines, for each temperature field $\hat{T}$, an energy problem $Z_{\hat{T}}$ which for a given divergence-free velocity field $\mathbf{u}$, gives the unique solution of the linearized energy equation. That is, $Z_{\hat{T}}$ is defined by

$$Z_{\hat{T}} : \mathbf{H}^1(\Omega) \to H^1(\Omega)$$

$$\mathbf{u} \mapsto Z_{\hat{T}}(\mathbf{u})$$

$$Z_{\hat{T}}(\mathbf{u}) = \text{unique solution of the linearized energy equation (53)} \tag{101}$$

$$\text{for a given solenoidal velocity field } \mathbf{u} \tag{102}$$

We shall prove, by means of a classical Banach fixed point theorem, that the following application is a contractive mapping:

$$\hat{T} \mapsto T = Z_{\hat{T}}(S(\hat{T})) \tag{103}$$

For this, we shall prove the existence of a positive constant $C<1$ such that $\|Z_{\hat{T}_1}(S(\hat{T}_1)) - Z_{\hat{T}_2}(S(\hat{T}_2))\|_{1,\Omega} < C\|\hat{T}_1 - \hat{T}_2\|_{1,\Omega}$. From this estimate, the existence of a fixed point is guaranteed, being the contractivity constant C dependent only on the given data.

The result will be established under two additional hypotheses:

[H1] The viscosity and thermal conductivity functions are Lipschitz continuous functions, with constants denoted by Lip(μ) and Lip(κ), respectively.

[H2] $\|S(\hat{T})\|_{1,\Omega} \leqslant C\|Z_{\hat{T}}(S(\hat{T}))\|_{\infty,\Omega}$.

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Hypothesis H1 matches for the viscosity models presented in the modelling section. We recall that in general, the least squares towards exponential correlations apply with good precision to moderate temperature ranges in liquids.

Hypothesis H2 is physically justified by the so-called Boussinessq approximation. With this foregoing simplification, the Navier–Stokes momentum equations are reduced to (4), where the density ρ_m is supposed to be linearly dependant with the temperature difference $\Delta T = T - T_m$. It is this variation that induces the transverse fluid motion. This strong hypothesis, only valid for small temperature differences ΔT, should be associated with small velocities, and for a steady laminar flow.

Hypothesis H2 is needed because *a priori* we have not an uniform estimate for the velocity solutions $S(\hat{T})$. Thus, from H2 we are assuming the continuity of the inverse application $Z_{\hat{T}}^{-1}$. Since the estimates of the solution $Z_{\hat{T}}(S(\hat{T}))$ are independent of $\hat{T}$ this hypothesis is not unrealistic (the forms $Z_{\hat{T}}(S(\hat{T}))$ are equicontinuous-like).

This assumption is certainly reasonable for small temperature differences as those encountered in free convection. In other way: small temperature differences will create small buoyancy effects and correlatively small velocity differences.

The main result is the following theorem.

Theorem 4.1
Let us assume that the hypotheses of Proposition 4.3 and the hypotheses H1 and H2 hold. Then, for small prescribed data (see (115)), the outer iteration $\hat{T} \mapsto T = Z_{\hat{T}}(S(\hat{T}))$ is a contractive mapping, which implies that the coupled problem (93)–(95) admits a solution.

Proof
For a given $\hat{T}$, let us recall the uncoupled problems

$$a_{\hat{T}}(T, \varphi) + b(\mathbf{u}, T, \varphi) + \frac{1}{2}\int_{\Gamma_N} [\mathbf{u}\cdot\mathbf{n}]^- T\varphi = 0 \quad \forall \varphi \in H^1_{0,\Gamma_D} \tag{104}$$

$$\mathbf{a}_{\hat{T}}(\mathbf{u}, \mathbf{v}) + \mathbf{b}(\mathbf{u}, \mathbf{u}, \mathbf{v}) + \frac{1}{2}\int_{\Gamma_N} [\mathbf{u}\cdot\mathbf{n}]^- \mathbf{u}\cdot\mathbf{v} - (\operatorname{div}\mathbf{v}, p) = \left(\frac{Gr}{Re^2}\hat{T}\mathbf{k} + \mathbf{F}, \mathbf{v}\right) \quad \forall \mathbf{v} \in \mathbf{H}^1_{0,\Gamma_D} \tag{105}$$

$$(\operatorname{div}\mathbf{u}, q) = 0 \quad \forall q \in L^2(\Omega) \tag{106}$$

Let $\hat{T}_1$ and $\hat{T}_2$ be given. According to definitions (99) and (100), the choice $Z_{\hat{T}_1}(S(\hat{T}_1)) - Z_{\hat{T}_2}(S(\hat{T}_2))$ is an admissible test function for the energy equation (104), and we have

$$\begin{aligned}
&\frac{1}{Pe}\int_\Omega \kappa(\hat{T}_1)\nabla Z_{\hat{T}_1}(S(\hat{T}_1))\nabla(Z_{\hat{T}_1}(S(\hat{T}_1)) - Z_{\hat{T}_2}(S(\hat{T}_2)))\\
&\quad -\frac{1}{Pe}\int_\Omega \kappa(\hat{T}_2)\nabla Z_{\hat{T}_2}(S(\hat{T}_2))\nabla(Z_{\hat{T}_1}(S(\hat{T}_1)) - Z_{\hat{T}_2}(S(\hat{T}_2)))\\
&\quad +\int_\Omega (S(\hat{T}_1)\cdot\nabla)Z_{\hat{T}_1}(S(\hat{T}_1))(Z_{\hat{T}_1}(S(\hat{T}_1)) - Z_{\hat{T}_2}(S(\hat{T}_2)))
\end{aligned}$$

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$$
\begin{aligned}
&-\int_\Omega (S(\hat{T}_2)\cdot\nabla)Z_{\hat{T}_2}(S(\hat{T}_2))(Z_{\hat{T}_1}(S(\hat{T}_1))-Z_{\hat{T}_2}(S(\hat{T}_2)))\\
&+\frac{1}{2}\int_{\Gamma_N}[S(\hat{T}_1)\cdot\mathbf{n}]^- Z_{\hat{T}_1}(S(\hat{T}_1))(Z_{\hat{T}_1}(S(\hat{T}_1))-Z_{\hat{T}_2}(S(\hat{T}_2)))\\
&-\frac{1}{2}\int_{\Gamma_N}[S(\hat{T}_2)\cdot\mathbf{n}]^- Z_{\hat{T}_2}(S(\hat{T}_2))(Z_{\hat{T}_1}(S(\hat{T}_1))-Z_{\hat{T}_2}(S(\hat{T}_2)))\\
&=0
\end{aligned} \tag{107}
$$

Re-arranging terms, we have

$$
\begin{aligned}
&\frac{1}{Pe}\int_\Omega \kappa(\hat{T}_1)\nabla(Z_{\hat{T}_1}(S(\hat{T}_1))-Z_{\hat{T}_2}(S(\hat{T}_2)))\nabla(Z_{\hat{T}_1}(S(\hat{T}_1))-Z_{\hat{T}_2}(S(\hat{T}_2)))\\
&\quad+\frac{1}{Pe}\int_\Omega (\kappa(\hat{T}_1)-\kappa(\hat{T}_2)\nabla Z_{\hat{T}_2}(S(\hat{T}_2))\nabla(Z_{\hat{T}_1}(S(\hat{T}_1))-Z_{\hat{T}_2}(S(\hat{T}_2)))\\
&\quad+\int_\Omega (S(\hat{T}_1)\cdot\nabla)(Z_{\hat{T}_1}(S(\hat{T}_1))-Z_{\hat{T}_2}(S(\hat{T}_2)))(Z_{\hat{T}_1}(S(\hat{T}_1))-Z_{\hat{T}_2}(S(\hat{T}_2)))\\
&\quad+\int_\Omega ((S(\hat{T}_1)-S(\hat{T}_2))\cdot\nabla)Z_{\hat{T}_2}(S(\hat{T}_2))(Z_{\hat{T}_1}(S(\hat{T}_1))-Z_{\hat{T}_2}(S(\hat{T}_2)))\\
&\quad+\frac{1}{2}\int_{\Gamma_N}[S(\hat{T}_1)\cdot\mathbf{n}]^-(Z_{\hat{T}_1}(S(\hat{T}_1))-Z_{\hat{T}_2}(S(\hat{T}_2)))(Z_{\hat{T}_1}(S(\hat{T}_1))-Z_{\hat{T}_2}(S(\hat{T}_2)))\\
&\quad+\frac{1}{2}\int_{\Gamma_N}([S(\hat{T}_1)\cdot\mathbf{n}]^- -[S(\hat{T}_2)\cdot\mathbf{n}]^-)Z_{\hat{T}_2}(S(\hat{T}_2))(Z_{\hat{T}_1}(S(\hat{T}_1))-Z_{\hat{T}_2}(S(\hat{T}_2)))\\
&=0
\end{aligned} \tag{108}
$$

Taking into account (57), hypothesis H1 and the bounds for the solutions $Z_{\hat{T}_1}(S(\hat{T}_1))$, $Z_{\hat{T}_2}(S(\hat{T}_2))$, $S(\hat{T}_1)$ and $S(\hat{T}_2)$, we have

$$
\begin{aligned}
&\frac{\kappa_1}{Pe}\|Z_{\hat{T}_1}(S(\hat{T}_1))-Z_{\hat{T}_2}(S(\hat{T}_2))\|_1^2\\
&\leqslant C\frac{Lip(\kappa)}{Pe}\|\hat{T}_1-\hat{T}_2\|_1\|Z_{\hat{T}_1}(S(\hat{T}_1))\|_\infty\|Z_{\hat{T}_1}(S(\hat{T}_1))-Z_{\hat{T}_2}(S(\hat{T}_2))\|_1\\
&\quad+\|S(\hat{T}_1)-S(\hat{T}_2)\|_{1,\Omega}\|Z_{\hat{T}_1}(S(\hat{T}_1))-Z_{\hat{T}_2}(S(\hat{T}_2))\|_{1,\Omega}\|Z_{\hat{T}_2}(S(\hat{T}_2))\|_{\infty,\Omega}\\
&\quad+\tilde{C}\|Z_{\hat{T}_2}(S(\hat{T}_2))\|_{\infty,\Omega}\|S(\hat{T}_1)-S(\hat{T}_2)\|_{1,\Omega}\|Z_{\hat{T}_1}(S(\hat{T}_1))-Z_{\hat{T}_2}(S(\hat{T}_2))\|_{1,\Omega}
\end{aligned} \tag{109}
$$

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Because the uniform bound for $Z_{\hat{T}_2}(S(\hat{T}_2))$ given by (97), we obtain

$$\|Z_{\hat{T}_1}(S(\hat{T}_1)) - Z_{\hat{T}_2}(S(\hat{T}_2))\|_{1,\Omega} \leqslant C\left(\frac{Lip(\kappa)}{\kappa_1}\|\hat{T}_1 - \hat{T}_2\|_{1,\Omega} + \|S(\hat{T}_1) - S(\hat{T}_2)\|_{1,\Omega}\right) \tag{110}$$

In the same way, working with the Navier–Stokes equations (105), we have that the choice $(S(\hat{T}_1) - S(\hat{T}_2))$ is an admissible test function. We have then

$$\begin{aligned}
&\frac{1}{Re}\int_\Omega \mu(\hat{T}_1) D(S(\hat{T}_1)) : D(S(\hat{T}_1) - S(\hat{T}_2)) - \frac{1}{Re}\int_\Omega \mu(\hat{T}_2) D(S(\hat{T}_2)) : D(S(\hat{T}_1) - S(\hat{T}_2))\\
&\quad + \int_\Omega (S(\hat{T}_1)\cdot\nabla) S(\hat{T}_1)\cdot(S(\hat{T}_1) - S(\hat{T}_2)) - \int_\Omega (S(\hat{T}_2)\cdot\nabla) S(\hat{T}_2)\cdot(S(\hat{T}_1) - S(\hat{T}_2))\\
&\quad + \frac{1}{2}\int_{\Gamma_N} [S(\hat{T}_1)\cdot\mathbf{n}]^- S(\hat{T}_1)\cdot(S(\hat{T}_1) - S(\hat{T}_2)) - \frac{1}{2}\int_{\Gamma_N} [S(\hat{T}_2)\cdot\mathbf{n}]^- S(\hat{T}_2)\cdot(S(\hat{T}_1) - S(\hat{T}_2))\\
&= \int_\Omega \frac{Gr}{Re^2}(\hat{T}_1 - \hat{T}_2)\mathbf{k}\cdot(S(\hat{T}_1) - S(\hat{T}_2))
\end{aligned} \tag{111}$$

Re-arranging terms, we have

$$\begin{aligned}
&\frac{1}{Re}\int_\Omega \mu(\hat{T}_1) D(S(\hat{T}_1) - S(\hat{T}_2)) : D(S(\hat{T}_1) - S(\hat{T}_2))\\
&\quad + \frac{1}{Re}\int_\Omega (\mu(\hat{T}_1) - \mu(\hat{T}_2) D(S(\hat{T}_2)) : D(S(\hat{T}_1) - S(\hat{T}_2))\\
&\quad + \int_\Omega (S(\hat{T}_1)\cdot\nabla)(S(\hat{T}_1) - S(\hat{T}_2))\cdot(S(\hat{T}_1) - S(\hat{T}_2))\\
&\quad + \int_\Omega ((S(\hat{T}_1) - S(\hat{T}_2))\cdot\nabla) S(\hat{T}_2)\cdot(S(\hat{T}_1) - S(\hat{T}_2))\\
&\quad + \frac{1}{2}\int_{\Gamma_N} [S(\hat{T}_1)\cdot\mathbf{n}]^-(S(\hat{T}_1) - S(\hat{T}_2))\cdot(S(\hat{T}_1) - S(\hat{T}_2))\\
&\quad + \frac{1}{2}\int_{\Gamma_N} ([S(\hat{T}_1)\cdot\mathbf{n}]^- - [S(\hat{T}_2)\cdot\mathbf{n}]^-) S(\hat{T}_2)\cdot(S(\hat{T}_1) - S(\hat{T}_2))\\
&= \int_\Omega \frac{Gr}{Re^2}(\hat{T}_1 - \hat{T}_2)\mathbf{k}\cdot(S(\hat{T}_1) - S(\hat{T}_2))
\end{aligned} \tag{112}$$

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Taking into account (57), hypotheses H1 and H2 and the bounds for the solutions $Z_{\hat{T}_1}(S(\hat{T}_1))$, $Z_{\hat{T}_2}(S(\hat{T}_2))$, $S(\hat{T}_1)$ and $S(\hat{T}_2)$, after grouping terms, we have

$$\frac{\mu_1}{Re}\|S(\hat{T}_1)-S(\hat{T}_2)\|_{1,\Omega}\leqslant \tilde{C}\left(\frac{\mu_1}{Re}-C\|Z_{\hat{T}_2}(S(\hat{T}_2))\|_{\infty,\Omega}\right)^{-1} \times\left(\frac{Lip(\mu)}{Re}\|\hat{T}_1-\hat{T}_2\|_{1,\Omega}\|Z_{\hat{T}_2}(S(\hat{T}_2))\|_{\infty,\Omega}+\frac{Gr}{Re^2}\|\hat{T}_1-\hat{T}_2\|_{1,\Omega}\right) \tag{113}$$

With (113) we go back to (110) and we obtain

$$\|Z_{\hat{T}_1}(S(\hat{T}_1))-Z_{\hat{T}_2}(S(\hat{T}_2))\|_{1,\Omega} \leqslant\tilde{C}\|\hat{T}_1-\hat{T}_2\|_{1,\Omega}\left(\frac{Lip(\kappa)}{\kappa_1}+\frac{Lip(\mu)}{Re}\|Z_{\hat{T}_2}(S(\hat{T}_2))\|_{\infty,\Omega}+\frac{Gr}{Re^2}\right) \tag{114}$$

Hence, if

$$\tilde{C}\left(\frac{Lip(\kappa)}{\kappa_1}+\frac{Lip(\mu)}{Re}\|Z_{\hat{T}_2}(S(\hat{T}_2))\|_{\infty,\Omega}+\frac{Gr}{Re^2}\right)<1 \tag{115}$$

then, (114) implies that the mapping Z is a contractive mapping (we recall that in the validity of the model, it is assumed that $Gr\ll Re^2$). □

4.5. A uniqueness result

In the finite-element analysis to be realized in the next part (see [38]), an uniqueness result is needed, in order to prove the convergence of the discrete approximations towards the unique solution of the coupled problem. As usual in Navier–Stokes equations, uniqueness will be obtained under small data (see the standard case in Theorem IV.2.2 of [3]).

With the set of hypotheses in the existence analysis of the previous section, we can state the following result.

Theorem 4.2

Under the hypotheses of the previous existence theorem (Theorem 4.1), the following coupled problem allows an unique solution for small prescribed data (see condition (126))

$$\mathbf{a}_T(\mathbf{u},\mathbf{v})+\tilde{\mathbf{b}}(\mathbf{u},\mathbf{u},\mathbf{v})-(\operatorname{div}\mathbf{v},p)=\left(\frac{Gr}{Re^2}T\mathbf{k}+\mathbf{F},\mathbf{v}\right)\quad\forall\mathbf{v}\in\mathbf{H}^1_{0,\Gamma_\mathrm{D}} \tag{116}$$

$$(\operatorname{div}\mathbf{u},q)=0\quad\forall q\in L^2(\Omega) \tag{117}$$

$$a_T(T,\varphi)+\tilde{b}(\mathbf{u},T,\varphi)=0\quad\forall\varphi\in H^1_{0,\Gamma_\mathrm{D}} \tag{118}$$

Proof

We follow the same arguments than in the contractivity analysis. First of all, we note that this time we have uniform bounds for the solutions (as fixed points of the uncoupled problems), which

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depend only on the problem data. We note further that having these uniform bounds for $\|\mathbf{u}\|_{1,\Omega}$ and $\|T\|_{1,\Omega}$, we can deduce uniform bounds for $\|D(\mathbf{u})\|_{0,\Omega}$ and $\|\nabla T\|_{0,\Omega}$.

Let $(\mathbf{u}_1, p_1, T_1)$ and $(\mathbf{u}_2, p_2, T_2)$ be two solutions of the coupled problem (116)–(118). If we take as test functions, respectively, $(\mathbf{u}_1-\mathbf{u}_2)$, (p_1-p_2) and (T_1-T_2) in the difference of the equations verified by each solution, we obtain

$$\begin{aligned}
&\frac{1}{Re}\int_\Omega \mu(T_1)D(\mathbf{u}_1):D(\mathbf{u}_1-\mathbf{u}_2)-\frac{1}{Re}\int_\Omega \mu(T_2)D(\mathbf{u}_2):D(\mathbf{u}_1-\mathbf{u}_2)\\
&+\int_\Omega(\mathbf{u}_1\cdot\nabla)\mathbf{u}_1\cdot(\mathbf{u}_1-\mathbf{u}_2)-\int_\Omega(\mathbf{u}_2\cdot\nabla)\mathbf{u}_2\cdot(\mathbf{u}_1-\mathbf{u}_2)\\
&+\frac{1}{2}\int_{\Gamma_N}[\mathbf{u}_1\cdot\mathbf{n}]^-\mathbf{u}_1\cdot(\mathbf{u}_1-\mathbf{u}_2)-\frac{1}{2}\int_{\Gamma_N}[\mathbf{u}_2\cdot\mathbf{n}]^-\mathbf{u}_2\cdot(\mathbf{u}_1-\mathbf{u}_2)\\
&+\int_\Omega(p_1-p_2)\,\mathrm{div}\,(\mathbf{u}_1-\mathbf{u}_2)=\int_\Omega\frac{Gr}{Re^2}(T_1-T_2)\mathbf{k}\cdot(\mathbf{u}_1-\mathbf{u}_2)
\end{aligned}\tag{119}$$

$$\int_\Omega(p_1-p_2)\,\mathrm{div}\,(\mathbf{u}_1-\mathbf{u}_2)=0\tag{120}$$

$$\begin{aligned}
&\frac{1}{Pe}\int_\Omega\kappa(T_1)\nabla T_1\nabla(T_1-T_2)-\frac{1}{Pe}\int_\Omega\kappa(T_2)\nabla T_2\nabla(T_1-T_2)+\int_\Omega(\mathbf{u}_1\cdot\nabla)T_1(T_1-T_2)\\
&-\int_\Omega(\mathbf{u}_2\cdot\nabla)T_2(T_1-T_2)+\frac{1}{2}\int_{\Gamma_N}[\mathbf{u}_1\cdot\mathbf{n}]^-T_1(T_1-T_2)\\
&-\frac{1}{2}\int_{\Gamma_N}[\mathbf{u}_2\cdot\mathbf{n}]^-T_2(T_1-T_2)=0
\end{aligned}\tag{121}$$

By following the same re-arrangement from (112) for the momentum equations, and taking into account (120), we obtain the following estimate (here, K_1 stands for the uniform bound of $\|\mathbf{u}_2\|_{1,\Omega}$)

$$\begin{aligned}
\frac{\mu_1}{Re}\|\mathbf{u}_1-\mathbf{u}_2\|_{1,\Omega}^2\leqslant C\Bigg(&\frac{Lip(\mu)}{Re}\|T_1-T_2\|_{1,\Omega}K_1\|\mathbf{u}_1-\mathbf{u}_2\|_{1,\Omega}\\
&+\|\mathbf{u}_1-\mathbf{u}_2\|_{1,\Omega}^2\|\mathbf{u}_2\|_{1,\Omega}+\frac{Gr}{Re^2}\|T_1-T_2\|_{1,\Omega}\|\mathbf{u}_1-\mathbf{u}_2\|_{1,\Omega}\Bigg)
\end{aligned}\tag{122}$$

We note in (122) that the term $\|\mathbf{u}_1-\mathbf{u}_2\|_{1,\Omega}$ appears squared or multiplied by $\|T_1-T_2\|_{1,\Omega}$.

In a similar way, referring as K_2 the uniform bound of $\|\nabla T_2\|_{0,\Omega}$, we obtain for the energy equation (121)

$$\frac{\kappa_1}{Pe}\|T_1-T_2\|_{1,\Omega}^2\leqslant C\left(\frac{Lip(\kappa)}{Pe}\|T_1-T_2\|_{1,\Omega}^2K_2+\|\mathbf{u}_1-\mathbf{u}_2\|_{1,\Omega}K_2\|T_1-T_2\|_{1,\Omega}\right)\tag{123}$$

Thus, we have from (123) that

$$\|T_1-T_2\|_{1,\Omega}\leqslant C\|\mathbf{u}_1-\mathbf{u}_2\|_{1,\Omega}\tag{124}$$

Taking into account this relation (124), we go back to estimation (122) and we have

$$\left(\frac{\mu_1}{Re} - \tilde{C}\left(\frac{Lip(\mu)}{Re} - 1 - \frac{Gr}{Re^2}\right)\right)\|\mathbf{u}_1 - \mathbf{u}_2\|^2_{1,\Omega} \leqslant 0 \tag{125}$$

Hence, if

$$\left(\frac{\mu_1}{Re} - \tilde{C}\left(\frac{Lip(\mu)}{Re} - 1 - \frac{Gr}{Re^2}\right)\right) > 0 \tag{126}$$

then, the coefficient which multiplies the term $\|\mathbf{u}_1 - \mathbf{u}_2\|^2_{1,\Omega}$ remains positive, we obtain $\mathbf{u}_1 = \mathbf{u}_2$ and then $T_1 = T_2$. The last conclusion for the temperature is due to (124).

Having $T_1 = T_2$ and $\mathbf{u}_1 = \mathbf{u}_2$ we go back to weak momentum equation (119) and we have in a distributional sense that $p_1 - p_2 = C$, but the outflow BCs with $T_1 = T_2$ and $\mathbf{u}_1 = \mathbf{u}_2$ makes that $C = 0$, so $p_1 = p_2$. □

Remark 4

From the study realized in this section, it follows that for the model with generalized outflow BCs analysed, if we consider in addition $\alpha, \beta \in \mathbb{R}^+$ and $g \in L^2(\Omega)$, the following problem is well posed: *find* $(\mathbf{u}, p, T) \in \mathbf{H}^1(\Omega) \times L^2(\Omega) \times H^1(\Omega)$ *such that, for all* $(\mathbf{v}, \varphi, \psi) \in \mathbf{H}^1_{0,\Gamma_D} \times L^2(\Omega) \times H^1_{0,\Gamma_D}$

$$\begin{gathered}
\alpha(\mathbf{u}, \mathbf{v}) + \mathbf{a}_T(\mathbf{u}, \mathbf{v}) + \mathbf{b}(\mathbf{u}, \mathbf{u}, \mathbf{v}) + \frac{1}{2}\int_{\Gamma_N} [\mathbf{u}\cdot\mathbf{n}]^- \mathbf{u}\cdot\mathbf{v} + (\operatorname{div}\mathbf{v}, p) = (\mathbf{F}, \mathbf{v}) + \frac{Gr}{Re^2}(T\mathbf{k}, \mathbf{v}) \\
(\operatorname{div}\mathbf{u}, \varphi) = 0 \\
\beta(T, \psi) + a_T(T, \psi) + b(\mathbf{u}, T, \psi) + \frac{1}{2}\int_{\Gamma_N} [\mathbf{u}\cdot\mathbf{n}]^- T\psi = (g, \psi)
\end{gathered} \tag{127}$$

All the previous proofs are easily adapted to this new situation. In consequence, one can show that (127) admits also a unique solution (under the corresponding newer *a priori* bounds and a new (and weaker) ellipticity condition instead of (25)).

This problem (127) arises when the corresponding evolution problem is discretized by backward difference formulas (see [25]).

5. CONCLUDING REMARKS

We have analysed in this first part a mathematical model associated with a flow into a channel, which considers a coupling between the steady Navier–Stokes equations and the scalar energy equation by taking into account the temperature dependence of the three thermophysical properties: dynamic viscosity, thermal conductivity and density. The mathematical model is stated as a generalized Boussinesq model, and some BCs are stated in the exit region. From this model, a closer variational formulation is stated, and this coupled problem was attacked by means of a fixed point strategy, which gives rise to uncoupled linear energy equation and the Navier–Stokes equations.

The variational formulation of the set of steady partial differential equations (30)–(32) state a problem which is not easy to handle, because there is no information available on the behaviour of the outflow term. As in Heywood *et al.* [10], we do not present an existence result for the original weak problem (30)–(32), which remains open and will be subject of future research. The analysis is far to be a straightforward generalization of the standard homogeneous Dirichlet techniques.

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We have been able, however, to propose and justify a variational formulation which is well adapted for the study of open flow situations and mixed convection regimes. Their mathematical analysis is easily adapted to other situations such as confined flow (for instance, the cavity problem). Existence and uniqueness results are possible to find, as usual, under small data hypothesis, for this kind of coupled problems. We recall that our main motivation for this work is that this kind of open flow problems are commonly found in real applications, such as chemical vapour deposition (see [39]).

The main difficulty of this coupled problem comes from the buoyant term, because if we do not consider the gravity (that is, taking $Gr=0$, as in [13]), we can obtain uniform bounds which are independent of the temperature element realizing the uncoupling, and from this, standard fixed point arguments applies (see [33]).

As long as the uniqueness result is stated, the next step consists of building an *approximate solution* for this problem towards a discretization procedure. Thus, in the next part of this work (see [38]), we shall continue with the analysis of a finite-element discretization of the proposed variational formulation, with numerical results which show that even if the predominant temperature effect in this model is given by the buoyancy, there exists an influence of the viscosity variations with the temperature. This influence becomes of great interest in the study of the associated convective heat transfer process (see [25]).

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